A Philosophical History of Infinitesimals

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Abstract: We explore the issue of providing a foundational framework for Leibnizian infinitesimals in the light of modern standard and nonstandard approaches. We outline a trichotomy of ordinals, cardinals and ringinals as a historiographic tool. A *ringinal* is a concept of infinite number, arithmetic in nature, different from Cantor's transfinite ordinals and cardinals. The continuum is not necessarily identifiable with $\mathbb{R}$;

even if one seeks such an identification, infinitesimals are not ruled out. Analysis with unlimited numbers (via the predicate *standard*) is possible in a conservative extension of Zermelo-Fraenkel set theory and in this sense is epistemologically 'safe.' We sketch a recent theory of infinitesimal analysis that formalizes Leibnizian definitions and heuristic principles while eschewing both the axiom of choice and ultrafilters, thus challenging received philosophical views on the nature of infinitesimals.

Keywords: infinitesimals, inassignables, law of continuity, Cantor, Leibniz.

Historia filosófica de los infinitesimales

Resumen: Investigamos la cuestión de proveer una estructura fundacional de los infinitesimales leibnizianos a la luz de los enfoques modernos estándares y no estándares. Resumimos una tricotomía de ordinales, cardinales y "ringinales" como una herramienta historiográfica. Un ringinal es un concepto de número infinito, aritmético en carácter, diferente de los ordinales y cardinales transfinitos de Cantor. El continuo no necesariamente se identifica con $\mathbb{R}$; incluso si se busca tal identificación, no se descartan los infinitesimales. El análisis con los números ilimitados (vía el predicado "estándar") es posible en una extensión conservadora de la teoría de conjuntos de Zermelo-Fraenkel y en este sentido es epistemológicamente "seguro". Esbozamos una teoría reciente de análisis infinitesimal que formaliza las definiciones leibnizianas y los principios heurísticos evitando el axioma de elección y los ultrafiltros, por consiguiente poniendo en duda las opiniones filosóficas recibidas sobre la naturaleza de los infinitesimales.

Palabras clave: infinitesimales, inasignables, ley de continuidad, Cantor, Leibniz.

1. Intermediate state between Leibniz and d'Alembert

Abraham Robinson included a sketch of the history of infinitesimal analysis in Chapter 10 of his book (Robinson 1966). Over half a century later, significant advances have taken place both in understanding this history and in axiomatic nonstandard analysis.

Indivisibles were controversial in the 17th century, not least in the eyes of the Jesuits who issued numerous bans against them (McCue 1968: 419;

Festa 1991: 101 and Festa 1992). Perhaps sensing the doctrinal burden of the (overly) evocative term *indivisible* (see Section 1.2), Leibniz coined the term *infinitesimal* in 1673 taking up a suggestion of Nicolaus Mercator.[1] Ever since the inception of infinitesimal calculus in the work of Leibniz and Newton, there have been attempts to delegitimize infinitesimals. Abraham Robinson claimed to have developed a formalisation of Leibnizian infinitesimals. Robinson's claim has been challenged by some historians and supported by others.

***1.1. Law of Continuity and* status transitus**

Leibniz formulated a number of heuristic principles that governed his infinitesimal calculus, such as the *Law of Continuity* (see e.g., Bos 1974; Katz and Sherry 2012). One of the formulations of the Law of Continuity involves the postulation of a *status transitus*, an intermediate stage before reaching the end of the process of vanishing (see Bair *et al.* 2021: 6.1). Such an intermediate stage witnesses the appearance of *inassignable* quantities such as infinitesimals. The Leibnizian distinction between assignable and inassignable quantity goes back to Nicholas of Cusa (1401-1464); see (Knobloch 2026: 5). A (positive) infinitesimal is smaller than every assignable number (see further in Section 4.3). Concerning such quantities, Leibniz wrote: "Although they are not assignable, they turn out to be something existing and not an absolute nothing."[2]

It is "a question of fundamental methodology" in the Leibnizian calculus that such infinitesimal quantities are fictional (cf. Archibald *et al.* 2024; Katz *et al.* 2024b). Another formulation of the Law of Continuity posits that "The rules of the finite are found to succeed in the infinite and vice versa."[3]

Leibniz's dealing in such notions was not uniformly accepted by his contemporaries.[4] One of the opponents was Michel Rolle, who initiated a

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[1] See Probst (2018: 200), who also notes that Wallis used the term *pars infinitesima* already in 1670.

[2] "Bien qu'elles ne soient pas assignables, elles se trouvent être quelque chose d'existant et non pas un rien absolu." (Leibniz as quoted in Bella 2019: 195).

[3] "[I]l se trouve que les regles du fini reussissent dans l'infini, […] et que vice versa les regles de l'infini reussissent dans le fini" (Leibniz 1702: 93–94) (original spelling preserved). Cf. Robinson (1966: 266). See further in Section 3.3.

[4] Today, we possess a better appreciation of Leibniz's work, and few scholars would accept Whiteside's claim that "very few proofs of any kind in classical mathematics will be allowable, and certainly none were given in the 17th century on any but the most elementary numerical level" (Whiteside 1961: 184).

lively debate at the French Academy (Mancosu 1989; Bair *et al.* 2018a: 2.9).

The 17th century acrimonious debates over indivisibles set the tone for centuries to come. In 18th century France, Jean D'Alembert reasoned as follows: "A quantity is either something or nothing: if it is something, it has not yet vanished; if it is nothing, it has literally vanished. The supposition that there is an intermediate state between these two is a *chimera*."[5] D'Alembert's attitude toward the intermediate state appears to be less positive than Leibniz's. Some historians today adopt the attitude of the former rather than the latter; see e.g., Section 1.4.

1.2. Theological context

Indivisibles and atomism were thought contrary to catholic doctrine as established by the Council of Trent in the 16th century, and specifically Session 13, canon 2. The said canon concludes that whoever denies the transubstantiation interpretation of the Eucharist, "let him be anathema". Opposition to atomism within Catholicism goes back at least to Duns Scotus in the 13th century (Cross 1998: 118). Theological concerns may have been behind Fermat's caution in presenting his method of *adequality* and the variable *E* (Katz *et al.* 2013; Bair *et al.* 2018b).

Leibniz was keenly aware of such doctrinal tensions over indivisibles. Already in 1668, he published a work on substance aimed: "to effect a reconciliation between Roman Catholics and Protestants. [...] These works are especially valuable for what they reveal about the motivations behind Leibniz's first account of substance" (Mercer and Sleigh 1994: 68).

Decades later, writing to des Bosses on 8 september 1709, Leibniz explicitly distanced himself from both transubstantiation and consubstantiation, and sketched a monad-based approach.[6]

1.3. Ordinals, cardinals, ringinals

Some historians today tend to interpret the thrust of their training in naive set theory as entailing that there could exist only two types of infinite

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[5] D'Alembert as translated in Boyer (1959: 248); emphasis added. See Lamandé (2019: 52) for the original.

[6] See Look and Rutherford (transl.) in Leibniz (2007: 153). On the theological background, see further in Katz *et al.* (2024b).

number: *ordinal* and *cardinal*; for examples, see Sections 1.4 and 1.5.[7] Understandably, they are puzzled by the appearance of infinite numbers in modern frameworks for analysis with infinitesimals (see Section 3).

Unlike ordinals and cardinals, such infinite numbers are naturally (nonstandard) elements of the *ring* of integers, or the semiring of natural numbers. Accordingly, such infinite numbers could be referred to as *ringinals*,[8] to emphasize the contrast with Cantorian infinities. Such a ringinal is necessarily greater than every naive counting number 1,2,3,… Laugwitz observed that Cantor had opposed such entities:

> [C]onvinced that he had discovered the one and only way to establish the actual infinite in mathematics after millenia of philosophical prejudices, Cantor was irritated by attempts of others to grasp the infinitely large, or to revive the infinitely small (Laugwitz 2002: 102).

Our distinction between Cantorian infinities and ringinals is related to Laugwitz's distinction "between Cantor's transfinite arithmetic and the theory of ordered algebraic structures." (Laugwitz 2002: 102). As noted by Laugwitz, "To Stolz, Veronese and Levi-Civita we owe early insights in ordered mathematical structures as a *third* aspect of the number concept" (Laugwitz 2002: 103; emphasis added). The three aspects alluded to by Laugwitz are, in our terminology, ordinal, cardinal, and ringinal.

1.4. Leibniz's bounded infinity and Knobloch's cardinals

With regard to Cantor's denial of the possibility of ringinals (see Section 1.3), some modern historians appear to follow suit. As a case study, we analyze Knobloch's comments on Leibnizian infinitesimal and infinite numbers, made in the context of an analysis of *Propositio XI* of *De Quadratura Arithmetica*; see e.g., Leibniz (2004). In the Leibnizian passage, λ denotes the curve admitting the axis of the ordinates as an asymptote,[9]

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[7] For a discussion of teleology-prone aspects of modern historiography of mathematics, see Bair *et al*. (2022: 4).

[8] Note that *ringinal* is a different concept from *numerosity*. The existence of a numerosity function is independent of the axioms of ZFC (Benci and Di Nasso 2003: Abstract), whereas the existence of ringinals is provable in ZFC and is, moreover, conservative over ZF; Hrbacek and Katz (2021).

[9] In the context of *Propositio XI*, the curve is of type $y = \frac{1}{x^2+1}$ or more precisely $x = \frac{1}{y^2+1}$. In Leibniz's figures, the axis of the ordinates is horizontal.

whereas (μ) denotes an infinitesimal abscissa, and $(\mu)\lambda$ denotes the corresponding ordinate which is necessarily infinite.[10] In this connection, Knobloch writes:

1. Leibniz uses also another terminology: *ordinata* $(\mu)\lambda$ *erit longitudine infinita, major qualibet assignabili* $_4B\ _4D$ (the ordinate will be of infinite length, larger than any assignable ordinate $_4B\ _4D$)[11]…
2. Hence there is an unavoidable consequence. The set of all finite cardinal numbers 1,2,3,… is a *transfinite set*. Its cardinal number is Alef$_0$.[12] This is the least cardinal number being [sic] larger than any finite cardinal number.
3. Leibniz's terminology implies actual infinity though he rejects the existence of an infinite number, etc.[13]

Knobloch's analysis starts (part 1) and ends (part 3) with Leibniz, suggesting that the intermediate discussion (part 2) of "transfinite sets" and Alef$_0$ is meant to shed light on Leibniz's mathematics. Yet the reader may well wonder what transfinite sets and the Cantorian Alef$_0$ have to do with Leibniz's mathematics.

Note the symbol $(\mu)\lambda$ used in Knobloch's passage. Leibniz exploited this symbol to denote a bounded infinity (*infinitum terminatum*); see further in Section 4.1. It is instructive to analyze Knobloch's reasoning in this pasage. Knobloch appears to argue that since Leibniz's bounded infinity $(\mu)\lambda$ is a magnitude greater than the naive integers 1,2,3,…, the "unavoidable consequence" would be that it must be at least Alef$_0$. But is such a consequence unavoidable? Besides the fact that Knobloch's argumentation would be meaningless to Leibniz who holds infinite wholes (such as the Cantorian Alef$_0$) to be contradictory because contrary to the part-whole principle, Knobloch's argument depends on an implicit assumption that any magnitude would necessarily be below some naive counting number 1,2,3,… In effect, Knobloch is denying the possibility of ringinals.

Knobloch claims that "Leibniz's terminology implies *actual infinity* though he rejects the existence of an infinite number" (emphasis added) in the passage above. But which "actual infinity" is Knobloch referring to exactly? Here Knobloch similarly fails to distinguish between infinite magnitude (namely, the Leibnizian *infinita terminata*) and infinite multitude

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[10] I.e., an *infinitum terminatum*, or in modern terms *unlimited*; see Section 3.2.

[11] We use *subscripts* on the left side, as in $_4B$, to make the formulas easier to read for the modern reader. In his manuscripts, Leibniz placed them on the same line as the letter, as in 4*B*.

[12] Here Alef$_0$ is Knobloch's notation.

[13] Knobloch (2018: 13-14; emphasis on "transfinite set" and numerals 1, 2, 3 added).

(namely, infinite wholes which Leibniz indeed rejected as contradictory).

Knobloch's interpretation of Leibniz ultimately leads him to attribute contradictions to Leibniz where there may be none. As analyzed in Katz *et al.* (2023), there is no contradiction between Theorems 11 and 45 in Leibniz's *De Quadratura Arithmetica*, as they deal with different notions of infinity: Theorem 11 deals with the *infinitum terminatum*, whereas Theorem 45 deals with an ideal perspective point at infinity. In a recent text, Knobloch discusses both theorems and concludes as follows:

> Here [i.e., in Theorem 45], Leibniz says the opposite of what he had said in the demonstration of theorem 11. There, he had explicitly excluded the possibility that the curve meets the asymptote in accordance with the meaning of its name (Knobloch 2024: 9).

It emerges that Knobloch prefers to attribute a contradiction to Leibniz rather than accept an interpretation differing from Ishiguro's as developed in Ishiguro (1990: Chapter 5). [14]

Meanwhile, Leibniz made it clear that his magnitudes violated the concept of comparability expressed in Euclid V, Definition 4, closely related to the Archimedean principle. This enabled Leibniz to use bounded infinities while adhering to the part-whole principle, for details see Section 4.3.

1.5. Sonar on hornangles

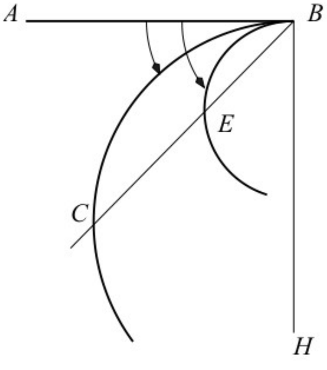

Figure 1. Sonar's analysis of hornangles

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[14] Knobloch adds the following sentence: "We shall come back to this matter of fact" (ibid.), but he never does.

Related misconceptions about non-Archimedean behavior occur in Sonar's work. Sonar analyzes de Saint Vincent's discussion of hornangles denoted $\measuredangle ABC$ and $\measuredangle ABE$ (Sonar 2021: 227). With reference to Figure 1, he shows that the hornangles must be necessarily smaller than a rectilinear angle $\measuredangle ABH$ as well as smaller than its fractions:

$$\frac{\measuredangle ABH}{2} > \frac{\measuredangle ABH}{2^2} > \cdots > \frac{\measuredangle ABH}{2^n} > \cdots > \measuredangle ABC \quad (S1)$$

However, the conclusion Sonar draws at this point is revealing of his assumptions, as he writes on page 228:

> But if *n* increases $\measuredangle ABH/2^n$ decreases and *hence it becomes clear* that
>
> $$\measuredangle ABC = \measuredangle ABE = 0 \quad (S2)$$
>
> holds for the horn angles. It is rather fascinating that Grégoire de Saint-Vincent realised this fact but that he could never accept it! (Sonar 2021: 228; emphasis added).

On page 228, Sonar apparently considers himself to have proved the claim that hornangles are zero, refers to such a claim as a "fact", and comments that de Saint Vincent never accepted such a fact. Note that de Saint Vincent's teacher Clavius was favorably disposed toward hornangles and was involved in a controversy with Peletier on this topic.[15] As acknowledged by Sonar: "Gottfried Wilhelm Leibniz will resume de Saint-Vincent's work later" (Sonar 2021: 228).

Indeed, Leibniz also used hornangles.[16] As late as 1826, Cauchy spoke of hornangles (*angles de contingence*) in his work on differential geometry (Cauchy 1826; see Bair *et al.* 2017a; Bascelli *et al.* 2018; Bair *et al.* 2019; Katz 2021; as well as Bair *et al.* 2022).

Thus, Sonar's argument for the vanishing of the hornangle seems to be based on a hidden hypothesis. Indeed, Sonar's inference from inequality (S1) to identity (S2) (described by Sonar's phrase "hence it becomes clear that") implicitly relies upon a property equivalent to the Archimedean axiom: if the hornangle $\measuredangle ABC$ satisfies the bound $\measuredangle ABC < 1/n$ for each naive integer n, then $\measuredangle ABC = 0$. What is therefore "clear" is that Sonar is relying on

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[15] Maierù (1990); Malet (1997); Axworthy (2018: 12–13) and references therein.

[16] For a discussion of hornangles in Leibniz, see, e.g., Katz *et al.* (2024b: 4.5).

the Archimedean axiom. If one refrains from relying upon the Archimedean axiom, there is no reason to assume that hornangles vanish. Sonar's inference is another instance of a denial of the possibility of ringinals (cf. Section 1.4).

Earlier in his book, Sonar writes on page 40:

> [A]lready Eudoxus knew that also other number systems – so-called non-Archimedean number systems – were conceivable [...] Such a system of quantities which was already known to the Greeks were cornicular angles or horn angles (Sonar 2021: 40).

Accordingly, Sonar is prepared to envision hornangles as part of a non-Archimedean number system. It is unclear how his comment on page 40 is to be squared with his comments on page 228 where hornangles are declared to provably vanish. Sonar's claim that "de Saint-Vincent realized [that hornangles vanish] but could never accept it" must be rejected as a presentist interpretation.

We examine an additional case of presentist treatment of historical infinitesimalists in Section 1.6.

1.6. Cauchy's infinitesimal delta function

In 1982, Lützen discusses delta functions that occurred in Poisson and Cauchy in connection with Fourier's integral theorem. Specifically, he mentions what is called today the Cauchy probability density. Lützen writes:

> They both used arguments which in distribution language would be (Lützen 1982: 115).

$$\frac{1}{2}\,\frac{\alpha}{\alpha^2+(x-a)^2} \to \delta(x-a), \quad \text{for } \alpha \to 0.$$

Note that the meaning of the first arrow would require explanation "in distribution language", which Cauchy did not possess. Lützen's formulation is rather presentist, since he does not even mention that α was actually infinitesimal for Cauchy.

As early as 1971, Hans Freudenthal briefly mentioned Cauchy's work on "singular integrals (i.e., integrals of infinitely large functions over infinitely small paths [δ functions])" (Freudenthal 1971: 135).

A 1989 article by Detlef Laugwitz contains a section 5.5 starting on page 227 entitled "Cauchy and delta functions". Laugwitz mentions Cauchy's use of the "language of infinitesimals" (Laugwitz 1989: 229) on page

289 of Note XVIII in Cauchy's publication (Cauchy 1827). This is in reference to Cauchy's infinitesimal α appearing in the expression

$$\frac{\alpha}{\alpha^2 + (x - a)^2}$$

which integrated against $F(x)$ produces the value $F(a)$ of F at the point a (up to a factor),[17] a property sometimes referred to as the sifting property (of the delta function). More specifically, Cauchy wrote:

> Moreover one finds, denoting by $\alpha, \in$ two infinitely small numbers,
>
> $$\frac{1}{2}\int_{a-\epsilon}^{a+\epsilon} F(\mu) \frac{\alpha d\mu}{\alpha^2 + (\mu - a)^2} = \frac{\pi}{2} F(a).$$

The source of the quotation is Cauchy (1827: 289).

Lützen is by no means the only author who acknowledges Cauchy's role in the prehistory of distributions while sweeping under the rug his use of infinitesimals in writing down his delta function. As early as 1955, van der Pol and Bremmer discuss the prehistory of distributions and mention Cauchy's role in the history of the delta function possessing what they refer to as the *sifting property* (van der Pol and Bremmer 1955: 64), but omit to mention Cauchy's use of infinitesimals in his delta function.[18]

2. Lagging formalisation

The formalisation and/or axiomatisation of analysis achieved during the decades around 1900 did not incorporate a formalisation of infinitesimals. The Zermelo–Fraenkel set theory (ZF) is a theory in the $\in$-language (thus, $\varnothing \in \{\varnothing\}$, etc.). The theory ZF was completely articulated by the 1920s. Meanwhile, an applicable formalisation of infinitesimals lagged by half a century (as we know in retrospect; see Section 3). Felix Klein sensed such a lag, and lamented it in the following terms:

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[17] From the modern point of view, the relation is not that of equality but rather infinite proximity. For further details, see Katz and Tall (2013).

[18] Van der Pol and Bremmer do quote Hermite on infinitesimals (van der Pol and Bremmer 1955: 62-63) and Poisson on infinitesimals (van der Pol and Bremmer 1955: 63-64).

> The question naturally arises whether […] it would be possible to *modify* the traditional foundations of infinitesimal calculus, so as to include actually *infinitely small* quantities in a way that would satisfy modern demands as to rigor; in other words, to construct a non-Archimedean system. The first and chief problem of this analysis would be to prove the mean-value theorem
>
> $$f(x+h)-f(x)=h\cdot f'(x+\vartheta h)$$
>
> [where $0 \leq \vartheta \leq 1$] from the assumed axioms. I will not say that progress in this direction is impossible, but it is true that none of the investigators have achieved anything positive (Klein 1908: 219; emphasis added).[19]

Progress in this direction started with Skolem (1933); see Section 3.1.

2.1. Campaign, from Russell to Carnap

In the meantime, a campaign of demonisation of infinitesimals took place, that was a thinly veiled attempt to conceal the failure of existing formalisations to incorporate infinitesimals; see e.g. Bair *et al.* (2013).

As detailed by Ehrlich (2006), Bertrand Russell and others publish texts claiming to derive the inconsistency of infinitesimals from philosophical principles, which merely embodied the philosophical prejudices of their authors.[20] According to Russell: "infinitesimals as explaining continuity must be regarded as unnecessary, erroneous, and self-contradictory" (Russell 1903: item 324).

It is hard to miss the tone of smug satisfaction in Florian Cajori's 1917 remarks concerning: *"wonderful* strides in the banishment of infinitely small quantities" (Cajori 1917: 152; emphasis added). Significantly, Cajori is of the (unsubstantiated) opinion that "Leibniz's philosophy of the calculus was poor." [21]

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[19] See also Kanovei *et al.* (2013: 6.1), Kanovei *et al.* (2018). Klein has not been treated fairly by all modern historians; see Bair *et al.* (2017c) and Heinig *et al.* (2023).

[20] A more sympathetic attitude toward infinitesimals at the time was displayed by the neo-Kantian school led by Hermann Cohen; see Laugwitz (2002: 111); Mormann and Katz (2013); Edgar (2020); Pringe (2023).

[21] Cajori (1917: 153). Cajori notes further that "[The] return to the use of infinitely small quantities is noticeable in several English texts of the second half of the [18th] century. An old lady once defended Calvinism by saying that if you took away her total depravity you took away her religion. There were mathematicians who believed that if you took away infinitely small quantities you took away all their mathematics" (Cajori 1917: 151). Cajori's

Rudolf Carnap highlights the purported force of one of his own philosophical innovations – the notion of a *pseudo concept* – as allegedly diagnosing the concept of the infinitesimal. To clarify further, Carnap includes a parenthetical epithet *empty words* (Carnap 1928: 306-307) for what Leibniz thought were *useful fictions*.[22]

Writing in *The Monist* in 1925 and possibly taking cue from Cantor's epidemiological metaphor of "cholera bacillus of mathematics" (Dauben 1990: 233 and 349, n. 55), Parkhurst and Kingsland speak of mathematics and metaphysics as territory "contaminated" (Parkhurst and Kingsland 1925: 633) and "infected" (Parkhurst and Kingsland 1925: 634) by infinitesimals, which risk causing "peripatetic fever" (Ibid.). Note that Cantor was writing in the midst of the fifth cholera outbreak (1881-1896), while Parkhurst and Kingsland, on the heels of the Spanish flu (1918-1920). The common denominator is the implied view of infinitesimals as a plague.

2.2. Courant's duty

Richard Courant elevates the *avoidance* of foggy, hazy infinitesimals to the status of a *duty*:

> [W]e must beware of regarding the derivative as the quotient of two quantities which are actually "infinitely small". [...] It is true that this fog that hung round the foundations of the new science did not prevent Leibnitz or his great successors from finding the right path. But this does not release us from *the duty of avoiding* every such hazy idea in our building-up of the differential and integral calculus (Courant [1937]1988: 101; emphasis added).

Courant does not attempt to explain how exactly Leibniz and "his great successors" managed to "find the right path" while dealing in foggy, hazy entities (but see Section 2.3). Courant's admonition is assorted with another *must* and is rather specific:

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suggestion of parallelism between a lady's 'depravity' and infinitesimals illustrates the attitudes toward infinitesimals prevalent during the 1910s.

[22] For details on useful fictions see Sherry and Katz (2012); Bair *et al.* (2021: 1.7); Katz *et al.* (2021).

> We must […] guard ourselves against thinking of *dx* as an “infinitely small quantity” or “infinitesimal”, or of the integral as the “sum of an infinite number of infinitely small quantities” (Courant [1937]1988: 80-81).

Echoing Carnap’s *empty words* (see Section 2.1), in 2021 Costantini devotes his Section 7 to an analysis of “The case of infinitesimals and other *empty notions*” (Costantini 2021: 285-286; emphasis added).

It apparently occurred to few of these 20th and 21st century scholars that, arguably, what we *must* develop is an appropriate formalisation of the Leibnizian distinction between assignable and inassignable quantities (see Sections 1.1 and 3.3).

2.3 Finding the right path

How *did* the founders of the calculus manage to “find the right path” as Courant put it (see Section 2.2)? Perhaps the entities in question were not as hazy as Courant made them out to be. Courant’s original comments date from 1927 (Courant 1927). Nearly half a century later, historian Bos will suggest that:

> A preliminary explanation of why the calculus could develop on the insecure foundation of the acceptance of infinitely small and infinitely large quantities is provided by the recently developed *non-standard analysis*, which shows that it is possible to remove the inconsistencies without removing the infinitesimals themselves (Bos 1974: 13).

However, Bos seems to walk back his “preliminary explanation” seventy pages later.[23]

2.4. Of darts and chimeras

In 1969, Bernstein and Wattenberg published a paper applying Robinson’s framework to measure theory. The paper opens with a discussion of the dart experiment (probabilistic analysis of throwing a dart at a target, Bernstein and Wattenberg 1969: 171). The dart paradox consists in the ob-

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[23] Bos (1974: 81-83). For an analysis of Bos’s remarks, see Katz and Sherry (2013: 11.3) and Bair *et al*. (2017b: 2.7).

servation that every point of the target has zero chance of being hit, yet some point does get hit by the dart. Bernstein and Wattenberg argued that Robinson's infinitesimal analysis enables a resolution of the paradox. They exploited a hyperfinite set containing ℝ;the resulting weighted counting measure assigns a nonzero infinitesimal probability to each real number.[24] Bernstein and Wattenberg presented the dart experiment as a feature of Robinson's approach.

In 1970, Alain Connes publishes a paper on ultrapowers and nonstandard analysis (Connes 1970) which cites Bernstein and Wattenberg (and even claims to improve on some of their results), indicating that Connes was familiar with their treatment of the dart paradox, *and appreciated it*.

By 2000, Connes seeks to present the dart experiment as a shortcoming of Robinson's framework (Connes 2000: 13-14). The critical comments by Connes were likely in response to Bernstein and Wattenberg.

Two decades later in 2021, Connes is still panning ultrafilters. He expresses the following two sentiments:

> [C1] a criticism of nonprincipal ultrafilters, and
> [C2] an endorsement of the Continuum Hypothesis as a tool for producing a Dixmier trace with desirable properties (Connes 2021: 47).

As a source for the construction of the desirable trace (using the "medial limit of Mokobodzki"),[25] Connes cites Meyer (1973). Here Meyer wrote: "Soit *E* l'ensemble de tous les ordinaux dénombrables: *E* a la puissance du continu (axiom du continu!), etc. (Meyer 1973: 200).

Note that this form of the Continuum Hypothesis implies, over ZF (see the beginning of Section 2), the existence of a well-ordering of the reals. This foundational material is exploited by Connes's source Meyer in the construction of the desirable trace. But such a well-ordering suffices to prove the existence of a nonprincipal ultrafilter, criticized in item [C1]. Therefore, the position as expressed by Connes in 2021 is mathematically incoherent.[26]

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[24] In axiomatic approaches to infinitesimal analysis, one exploits a finite set containing all standard real numbers. The existence of such a set is proved by the theory BSPT, which is a subtheory of IST and HST. The theory BSPT is conservative over ZF, as proved in Hrbacek and Katz (2021: 10). See further in Section 3.2.

[25] Note that Larson (2009) showed that ZFC does not prove the existence of medial limits.

[26] See further in Katz and Leichtnam (2013), Kanovei *et al.* (2015), Hrbacek and Katz (2021: 8.6), and Sanders (2018). A related critique of Robinson's framework by Paul Halmos is analyzed in Błaszczyk *et al.* (2016).

Furthermore, it turned out recently that nonstandard analysis can be practiced without ultrafilters.[27]

2.5. Emotionally charged

The industrialisation of the calculus programs starting in the 1960s produced a series of textbooks setting a certain classroom style. Advocates of infinitesimals argue that such classrooms create an emotionally charged atmosphere (foreshadowed by the tone of Courant's remarks quoted in Section 2.2):

> Students are often given emotionally charged instructions to avoid thinking of dy/dx as a quotient and to conceptualize it as a limit, even though the formulae of the calculus visibly seem to operate as if it is a quotient involving symbols that can be shifted around to change differential equations into integrals (Tall 2013: 336).

In such classrooms, instructors sometimes make deprecating comments at the expense of infinitesimals, inculcating in the student a faith that there is something fundamentally wrong with them. Students also tend to internalize a literal interpretation of the epithet *real* in the expression *real number*.[28]

3. From Fine Hall to modern infinitesimal analysis

Abraham Robinson (1918-1974) was an applied mathematician,[29] logician, and inventor of nonstandard analysis (NSA). Robinson named his theory:

> *Non-standard Analysis*[,] since it involves and was, in part, inspired by the so-called Non-standard models of Arithmetic whose existence was first pointed out by T. Skolem (Robinson 1966: vii; emphasis added).

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[27] See Section 3.2 on axiomatic frameworks such as SPOT.

[28] Studies of infinitesimal-based pedagogy include Katz and Polev (2017), Kuhlemann (2022), Kuhlemann (2024), Kuhlemann (2025), and Agnesi (2023).

[29] It is to be noted that Abraham Robinson's *Wing Theory* (Robinson and Laurmann 1956) is not a branch of model theory, but rather "an admirable compendium of the mathematical theories of the aerodynamics of aerofoils and wings" (Lighthill 1956).

Thus, the name of the field was influenced by Skolem's work.[30]

3.1. Ehrlich and Dauben on NSA

As noted by Ehrlich,

> After mathematicians had been taught for decades that a consistent theory of the calculus based on infinitesimals was impossible, Abraham Robinson was certainly swimming against the tide when he proved otherwise (Ehrlich 2022).

Building upon earlier breakthroughs by Thoralf Skolem (1933), Edwin Hewitt (1948), Jerzy Łoś (1955), and others, Robinson first introduced NSA in a 1961 article (Robinson 1961)[31] and then in the 1966 book (Robinsons 1966).[32] Dauben notes that

> Robinson succeeded in showing the reasonableness of "redrawing" the early history of the calculus to reinstate past views that, cast in the light of nonstandard analysis, could be seen more clearly (Dauben 2021: 327).

As anticipated apprehensively by Courant (see Section 2.2), *dx* is infinitesimal,[33] and the definite integral is defined via the sum of a nonstandard

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[30] It is therefore historically inaccurate to claim, as Loeb and Wolff do, that "Robinson chose the name 'nonstandard analysis' because the nonstandard world is used to analyze the standard one" (Loeb and Wolff 2015: vii). As the passage we quoted from Robinson's book indicates, his choice of the name 'nonstandard analysis' was inspired by Skolem's models, not by the reason claimed by Loeb and Wolff, who go on to lodge the following philosophical claim: "Internal set theory, on the other hand, works with only the nonstandard world, but recognizes some elements of that world as being 'standard'" (ibid.). Loeb and Wolff use such a philosophical claim to buttress their contention concerning the alleged impossibility of carrying out certain constructions (nonstandard hulls, Loeb measures, etc.) in axiomatic approaches to NSA. They conclude: "These constructions do not make sense in internal set theory because there is no standard world" (ibid.). Their contention is incorrect, as shown in Hrbacek and Katz (2023a), where both nonstandard hulls and Loeb measures are handled within an axiomatic approach to NSA. On axiomatic approaches to NSA, see further in Section 3.2.

[31] Robinson reported that the idea of infinitesimal analysis came to him as he was walking into Fine Hall in the fall of 1960. This is recounted in Robinson's biography (Dauben 1995: 281).

[32] For an analysis of the relation between the infinitesimals of Leibniz and Robinson, see Bair *et al*. (2021).

[33] This is the case in Robinson's framework. Similar notation *dx* is used in traditional

number of infinitely small quantities,[34] challenging Russell's claim that "the so-called infinitesimal calculus […] has nothing to do with the infinitesimal" (Russell 1903: item 308).

3.2. Axiomatisations: IST, HST, SPOT; conservativity

In the mid-1970s, Edward Nelson (1977) developed an axiomatic, or syntactic, framework for NSA, called Internal Set Theory (IST). At the same time, Karel Hrbacek (1978) developed a different axiomatic/syntactic framework, now called HST; see the monograph Kanovei and Reeken (2004).

The theory SPOT (an acronym of its axioms) developed in Hrbacek and Katz (2021) is a subtheory of both IST and HST.

Whereas IST and HST are conservative extensions of ZFC (ZF plus the Axiom of Choice), the theory SPOT is a conservative extension of ZF itself.[35] It is a theory in the st-∈-language, where "st" is a one-place predicate (thus, st(x) reads "x is standard") formalizing the Leibnizian distinction between assignable and inassignable quantities.[36]

Since SPOT incorporates the axioms of ZF, the natural numbers N and the real numbers $\mathbb{R}$ are developed as usual. A number is *limited* if its absolute value is smaller than some standard real number; otherwise it is *unlimited*.[37] An infinitesimal is a number smaller in absolute value than every positive standard number.[38] Two numbers are called *infinitely close* if their difference is an infinitesimal.

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non-infinitesimal analysis, where this is not the case.

[34] Namely, the integral is the standard number infinitely close to such a sum. For further details, see e.g., the textbook Vakil (2011).

[35] This feature of SPOT enables a reverse-mathematical analysis of the role of the axiom of choice in mathematics with infinitesimals, something that was not possible over the classical axiomatic theories such as IST and HST that themselves rely on the axiom of choice.

[36] Note that in axiomatic set theories such as IST, HST and SPOT, the standard (or "naïve" as Reeb called them) integers do not exhaust the set $\mathbb{N}$. Standard integers do not form a set. The separation axiom schema of ZF still holds in SPOT but it only applies to ∈-formulas, not to st-∈-formulas involving the predicate *standard*. See further in Section 3.4.

[37] An unlimited element of $\mathbb{N}$ is what was referred to as a *ringinal* in Section 1.3.

[38] This definition of an infinitesimal is valid both in Robinson's original framework and in syntactic/axiomatic frameworks such as IST, HST, and SPOT. It formalizes Leibniz's definition of infinitesimal as smaller than every assignable number; see Section 1.1.

In addition to the axioms of ZF (which apply to all $\in$-formulas),[39] SPOT has the following three axioms. Here $\exists^{st}$ and $\forall^{st}$ denote quantification over standard entities only, and N is the set of natural numbers as defined in ZF (see above).

T (Transfer) Let ϕ be an $\in$-formula with standard parameters.
Then $\forall^{st} x\, \phi(x) \Leftrightarrow \forall x\, \phi(x)$.
O (Nontriviality) $\exists \nu \in \mathbb{N}\ \forall^{st} n \in \mathbb{N}\ (n \neq \nu)$.
SP (Standard Part) Every limited real is infinitely close to a unique standard real, called its *shadow*.
Equivalently,
SP' $\forall A \subseteq \mathbb{N}\ \exists^{st} B \subseteq \mathbb{N}\ \forall^{st} n \in \mathbb{N}\ (n \in B \Leftrightarrow n \in A)$.[40]

The Transfer axiom (schema) is a formalisation of the Leibnizian *Law of Continuity* (see Sections 1.1 and 3.3). For a more detailed introduction to SPOT and related theories such as SCOT and BSPT see recent work (Hrbacek and Katz 2021, 2023b, 2023c; Hrbacek 2024).

3.3. Leibniz's laws compared to the axioms of SPOT

The simplicity of the additional axioms of SPOT (compared to some of the more abstruse axioms of ZF) suggests that a formalisation of infinitesimals may not have been beyond the intellectual prowess of the pioneers of formalisation from over a century ago, which calls to mind the contingency of the historical evolution of mathematics.[41]

The viability of applying nonstandard analysis (or its axiomatic versions such as SPOT) to interpreting the procedures of the historical

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[39] As mentioned in note 36, the axiom schema of separation (comprehension) applies to all $\in$-formulas, but not to formulas involving the new predicate st, so that in particular the standard integers do not form a set. This feature has been used to provide an account of the sorites paradox in Dinis and van den Berg (2019); see also Dinis (2023).

[40] Intuitively, the equivalence of SP and SP' can be understood in terms of the binary expansion of a real number x . If A denotes the set of ranks at which the corresponding digit of x is 1, and B denotes the set of ranks at which the corresponding digit of its shadow sh(x) is 1, then the fact that x and sh(x) are infinitely close corresponds to the fact that A and B agree at all standard (limited) ranks. The detailed argument is slightly more technical due to the non-uniqueness of binary expansion; see Hrbacek and Katz (2021: Lemma 2.4).

[41] For a discussion of the issue of determinism vs contingency of the historical evolution of mathematics, see e.g., Mancosu (2009) and Bair *et al*. (2022: Section 4).

infinitesimalists depends crucially on the procedure/foundation distinction.[42]

In sum, the theory SPOT has three axioms (in addition to the ZF axioms): (1) **S**tandard **P**art, (2) n**O**ntriviality, and (3) **T**ransfer. We note the following.

1. Leibniz emphasized repeatedly that he is working with a relation of *equality up to* terms that need to be discarded, rather than with exact equality; he was aware of the need to take the assignable part (in modern terminology, Standard Part or shadow) of a quantity when appropriate, as in passing from $\frac{2x+dx}{a}$ to $\frac{2x}{a}$ in the calculation of $\frac{dy}{dx}$ when $ay=x^2$; see e.g., (Leibniz 1701).
2. Leibniz postulated inassignable infinitesimals, i.e., Nontriviality.
3. Leibniz had a Law of Continuity one of whose formulations was that the rules of the finite succeed in the infinite and vice versa, which Robinson described as being remarkably close to Transfer (Robinson 1966: 266; see also Section 3.3). Where Leibniz spoke of assignable and inassignable quantities, Robinson spoke of standard and nonstandard numbers.

Arguably, Berkeley's logical criticism applies to Leibniz's calculus if and only if it applies to SPOT; see further in Section 4 (see also Bair *et al.* 2022: 1.1).

3.4. A note on well-foundedness

Since the axioms of SPOT include the axioms of ZF (for ∈-formulas), the definitions of ℕ and ℝ carry over from ZF. For example, ℕ could be defined as the smallest inductive set, and ℝ by Dedekind cuts or Cauchy sequences. There is therefore no reason to use new symbols for the natural numbers or the reals. The same remark applies to all standard concepts (ordinals, cardinals, etc.). Some standard concepts (limit, derivative …) also have nonstandard definitions in terms of infinitesimals, which may have their advantages (e.g., in teaching).

The axiomatic approach to NSA does not proceed via models; it axiomatizes its universe of discourse – say, by the axioms of SPOT. The uni-

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[42] This point seems to have been overlooked by the authors of Archibald *et al.* (2024). See further in Bascelli *et al.* (2016), Bair *et al.* (2021).

verse of discourse of SPOT satisfies the axiom of foundation (i.e., every nonempty *set* has an ∈-minimal element). The additional predicate "st" enables descriptions of certain *non-sets* (classes) that the traditional mathematician ignores (such as the class of all nonstandard integers). The universe is not well-founded with respect to these non-sets.

Traditional Platonist mathematicians may find such a viewpoint difficult to accept because on their view every collection of elements of some set, no matter how described, should be a set. But the approach fits well with Hamkins's multiverse view of set theory. For example, Fletcher *et al.* (2017: 228-230, 7.3) discuss the relationship between nonstandard analysis and the Gitman–Hamkins "toy model", developed in Gitman and Hamkins (2010), of the set-theoretic multiverse (Hamkins 2011, 2012).

4. From infinite sets to ringinals and recent work

In this section, we will provide additional historical context for the idea of formalizing the procedures of the historical infinitesimal calculus in the theory SPOT summarized in Section 3.2.

4.1. Positing infinite sets

As is well known, the existence of an infinite set is an axiom of ZF as formalized in the 20th century by Zermelo. Positing infinite sets was a departure from an ancient philosophical tradition at least as old as Aristotle (see Ugaglia 2022) that held that only potential infinity is conceivable (see further in Bair *et al.* 2022: 1.3). Thus, in *Physica* one finds:

> It is reasonable that there should not be held to be an infinite in respect of addition such as to surpass every magnitude, but that there should be thought to be such an infinite in the direction of division. [...] It is reasonable too to suppose that in number there is a limit in the direction of the minimum, and that in the other direction every multitude is always surpassed. In magnitude, on the contrary, every magnitude is surpassed in the direction of smallness, while in the other direction there is no infinite magnitude. [...] in the direction of largeness it is always possible to think of a larger number; for the number of times a magnitude can be bisected is infinite. Hence this infinite is potential, never actual: the number of bisections that can be taken always surpasses any definite multitude. But this number is not separable, and its infinity does not

> persist but consists in a process of coming to be, like time and the number of time (*Physica* III 207b3–15).[43]

Infinite wholes in mathematics were outside the conceptual resources of both Leibniz and Berkeley; both of them subscribed to the idea that infinite wholes are contradictory. Specifically, Leibniz's analysis of Galileo's paradoxes led him to believe that infinite wholes contradict Euclid's part-whole principle (see e.g., Bair *et al.* 2022: 2.8).

In the Leibnizian calculus, one works with both infinitesimal and infinite numbers. Leibniz had a special term for the inverse of an infinitesimal. He referred to it as *infinitum terminatum* (literally: bounded infinity). The latter was contrasted with *infinitum interminatum* (unbounded infinity, such as an unbounded infinite line) (see Bair *et al.* 2021: 2.2). *Infinita terminata* were useful in geometry and the calculus, whereas *infinita interminata* were useless because contradictory (as mentioned above). Leibniz's bounded infinity operates as a kind of ringinal (see Section 1.3) in the procedures of the Leibnizian calculus.

4.2. Postulating predicates and infinite sets

As noted in Section 4.1, the modern framework ZF postulates that an infinite set exists. Somewhat analogously, one introduces a new predicate st and postulating its basic properties, in particular distinguishing between standard and nonstandard numbers, in modern axiomatic theories such as IST, HST, or SPOT (see Section 3). Both the axiom of infinity and modern infinitesimals are 20th century innovations. Arguably, the acceptance of infinite sets is neither more nor less justified than the acceptance of a richer background language incorporating the one-place relation st (in addition to the two-place relation $\in$). Infinite sets and the properties of the predicate st are similar in that both are *postulated* rather than *constructed*. Bedürftig and Murawski describe the somewhat paradoxical situation today as follows:

> It is astonishing that today everybody believes in the Axiom of Infinity generating infinitely large sets and infinite cardinal numbers but infinitely small quantities and infinitesimal numbers are sometimes indignantly rejected (Bedürftig and Murawski 2018: 201).

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[43] See e.g., https://classics.mit.edu/Aristotle/physics.3.iii.html.

Unlike infinite sets (which have almost no antecedent in mathematics before Cantor), the predicate "st" has a historical antecedent in Leibniz's distinction between assignable and inassignable quantities (see Section 1.1).

4.3. Leibniz on violation of Euclid V, Definition 4

Leibniz makes it clear that *dx* is an element of a non-Archimedean number system in a 1695 article in *Act. Erud.* (Leibniz 1695a: 322) in response to Nieuwentijt. Here Leibniz makes it clear that his incomparable infinitesimals violate the comparability notion put forward by Euclid in Book V Definition 4, closely related to the Archimedean property. Similar comments appeared in a letter to l'Hôpital (Leibniz 1685b: 288) the same year.[44] Notes Malet: "By the late 17th century […] the dividing line between numbers and continuous magnitudes was largely gone" (Malet 2012: 213). For details see Bair *et al.* (2018a: 3.2). Interpretation of the Leibnizian calculus is an area of lively debate; see e.g., Eklund (2020), Esquisabel and Raffo Quintana (2021), Katz *et al.* (2022), Archibald *et al.* (2024), Bair *et al.* (2023), Katz, Sherry and Ugaglia (2023a, 2023b), Katz, Kuhlemann and Sherry (2024a), Katz *et al.* (2024b), Ugaglia and Katz (2024), Knobloch (2024), Arthur and Rabouin (2024a, 2024b), Katz and Kuhlemann (2023), Katz (2025).

4.4. Which $\mathbb{R}$ provides a better background?

Can one exploit the predicate "st" before providing a construction of a non-Archimedean number system with standard and nonstandard elements? The key question here is what is meant by *construction*. Leibniz did not possess a modern set-theoretic construction of either a Weierstrassian real line $\mathbb{R}_W$, or a Hrbacek–Nelson type real line $\mathbb{R}_{HN}$ (see Section 3). Rather,

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[44] It is therefore difficult to agree with Gert Schubring's claim that "'the founders of the calculus' had not at all created it with a non-Archimedean continuum in mind" (Schubring 2022: 6), or that "Leibniz always refused to be identified with a foundation based on—rather vaguely conceived—infinitesimals" (Schubring 2022: 1). Contrary to Schubring's claim that our group "for several years has been leading a *crusade* against the historiography of mathematics, in particular in its old-fashioned forms" (ibid.; emphasis added), we are only "against" sloppy historical writing of the kind unfortunately found in Schubring's publications; see e.g., Błaszczyk *et al.* (2017b) and Katz *et al.* (2024b: nn. 46,50). Schubring's critique is analyzed in Bair *et al.* (2022: Sections 2–3).

Leibniz presented a collection of coherent mathematical procedures for infinitesimal calculus (exploiting the distinction between assignable and inassignable numbers), and one can ask: comparing theories ZF and SPOT (see Section 3.2), which one provides a better background for formalizing the kind of procedures Leibniz was working with?

Following the creation of modern infinitesimal analysis by Robinson in the 1960s, the field has developed into a vast research area that is not easy to survey meaningfully in an article on the history of philosophy. Many important applications are presented in the 2015 monograph edited by Loeb and Wolf (2015). Other recent exciting work includes Tao (2017), Sanders (2018), Dinis and van den Berg (2019), Goldbring (2022), Jin (2023), Hrbacek (2024).

5. Conclusion

Due to their apparently elusive nature, infinitesimals as practiced by Leibniz and others have historically been the subject of reservations ranging from criticism to outright claims of inherent contradiction. Some historians have seen modern theories of infinitesimals as developed by Robinson and others as a vindication of the work of classical infinitesimalists. Other historians have argued that modern infinitesimals exploit mathematical resources undreamt of by historical infinitesimalists.

The latter criticism is mitigated by means of the distinction, elaborated e.g., in Błaszczyk *et al.* (2017a) and Bair *et al.* (2021), between *foundations* and *procedures*. Namely, while the *foundational* aspects of modern infinitesimal frameworks may involve model theory, ultraproducts, axiom of choice, and other modern resources, the *procedures* of such theories (such as the relation of infinite proximity, use of ringinals and associated infinitesimal partitions) provide a closer fit for the procedures of the classical infinitesimalists than do the procedures of the traditional (non-infinitesimal) Weierstrassian framework.

A more far-reaching response is to exploit axiomatic approaches outlined in Section 3. A recent axiomatisation makes no use of the modern set-theoretic resources mentioned above; a conservativity result shows that it is possible to avoid any reliance on the axiom of choice or ultraproducts. The axioms involved find close parallels in heuristic principles explicit in Leibniz's *oeuvre*, such as his dichotomy of assignable and inassignable number and his Law of Continuity (see Sections 1.1 and 3.3).

Nonstandard analysis is sometimes criticized for not being sufficiently *effective* or for not being a satisfactory model for real world phenomena, due to its alleged reliance on the axiom of choice, whereas the elements of the so-called standard models N and R (constructible in ZF) seem to possess direct referents. Such objections have now become obsolete due to the existence of theories that are conservative extensions of ZF where these standard models also contain nonstandard/inassignable numbers (like unlimited numbers and infinitesimals), seen as *fictional* by Leibniz.

The existence of such modern formalisations of the Leibnizian calculus suggests that infinitesimals are "safer" epistemologically than is sometimes thought, and reveals the hidden potential of the notion of the continuum sometimes thought to have been definitively captured by the mathematical developments from a century and a half ago.[45]

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